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BOUNDEDNESS IN THE PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate bounds for solutions of the the perturbed functional differential systems.

1. Introduction

As is traditional in a pertubation theory of nonlinear differential systems, the behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. Among useful methods for investigating the qualitative behavior of the solutions of perturbed nonlinear system of differential systems, there are the method of variation of constants formula, Lyapunov' second method, and the use of inequalities. The theory of integral inequalities can be employed to study various properties of nonlinear differential systems. In the presence the method of integral inequalities is as efficient as the direct Lyapunov's method.

Pinto[13,14] introduced h-stability(hS) which is an important extention of the notions of exponential asymptotic stability and uniform Lipschitz stability. Also, he obtained some properties about asymptotic behavior of solutions of perturbed h-systems, some general results about asymptotic integration and gave some important examples in [14]. He obtained a general variational h-stability and some properties about asymptotic behavior of solutions of differential systems called h-systems. Also, he studied some general results about asymptotic integration and gave some important examples in [13]. Choi and Ryu [3], Choi, Koo[5], and Choi et al. [4] investigated bounds of solutions for the perturbed

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functional differential systems. Also, Goo [7,8,9,10] studied the boundedness of solutions for the perturbed functional differential systems.

In this paper, we investigate bounds of solutions of the perturbed functional differential systems.

2. Preliminaries

We consider the perturbed functional differential equation

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds, \; y(t_0) = y_0, \tag{2.1}$$

where $t \in \mathbb{R}^+ = [0, \infty)$, $x \in \mathbb{R}^n$,$f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t, 0) = 0$, the derivative $f_x \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0, 0) = 0$ and T is a continuous operator mapping from $C(\mathbb{R}^+, \mathbb{R}^n)$ into $C(\mathbb{R}^+, \mathbb{R}^n)$. The symbol $|\cdot|$ will be used to denote arbitrary vector norm in $\mathbb{R}^n$. We assume that for any two continuous functions $u, v \in C(I)$ where I is the closed interval, the operator T satisfies the following property:

$$u(t) \leq v(t), 0 \leq t \leq t_1, t_1 \in I,$$

implies $Tu(t) \leq Tv(t), 0 \leq t \leq t_1$, and $|Tu| \leq T|u|$.

Equation (2.1) can be considered as the perturbed equation of

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.2}$$

Let $x(t, t_0, x_0)$ be denoted by the unique solution of (2.2) passing through the point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ such that $x(t_0, t_0, x_0) = x_0$. Also, we can consider the associated variational systems around the zero solution of (2.2) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

We recall some notions of h-stability [13].

Definition 2.1. The system (2.2) (the zero solution $x = 0$ of (2.2)) is called an *h-system* if there exist a constant $c \geq 1$ and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 2.2. The system (2.2) (the zero solution $x = 0$ of (2.2)) is called (hS) *h-stable* if there exist $\delta > 0$ such that (2.2) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [6].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is *t_∞-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t), \; \dot{} = \frac{d}{dt} \tag{2.5}$$

for some $S(t) \in \mathcal{N}$.

We give some related properties that we need in the sequal.

Lemma 2.4. [14] *The linear system*

$$x' = A(t)x, \; x(t_0) = x_0, \tag{2.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\,h(t)\,h(t_0)^{-1} \tag{2.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (2.6).

We need Alekseev formula to compare between the solutions of (2.2) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \; y(t_0) = y_0, \tag{2.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.5. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

THEOREM 2.6. [3] *If the zero solution of (2.2) is hS, then the zero solution of (2.3) is hS.*

THEOREM 2.7. [4] *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.3) is hS, then the solution $z = 0$ of (2.4) is hS.*

LEMMA 2.8. [7] *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_2(s) (\int_{t_0}^{s} \lambda_3(\tau) u(\tau) d\tau) ds, \ 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau) ds\Big], \ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup\Big\{ t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau) ds \in \mathrm{domW}^{-1} \Big\}.$$

LEMMA 2.9. [7] *Let $u, p, q, w, r \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) w(u(\tau)) + v(\tau) \int_{t_0}^{\tau} r(a) u(a) da) d\tau) ds, \ t \geq t_0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da) d\tau) ds\Big], \ t_0 \leq t < b_1,$$

where W and W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup\Big\{ t \geq t_0 : W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da) d\tau) ds \in \mathrm{domW}^{-1} \Big\}.$$

3. Main results

In this section, we investigate the bounded property for the nonlinear functional differential systems.

THEOREM 3.1. *Let* $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in* u, $u \leq w(u)$, *and* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$. *Suppose that* $f_x(t, 0)$ *is* t_∞*-similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \geq t_0 \geq 0$ *and* $|x_0| \leq \delta$ *for some constant* $\delta > 0$, *the solution* $x = 0$ *of (2.2) is hS with the nondecreasing function* h, *and* g *in (2.1) satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| \leq a(s)w(|y(s)| + b(s)|Ty(s)|),\ t \geq t_0 \geq 0,$$

and

$$|Ty| \leq \int_{t_0}^{t} k(s)|y(s)|ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, *and* $\int_{t_0}^{\infty} k(s)ds < \infty$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (2.1) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) \int_{t_0}^{s} k(\tau)d\tau)ds\Big], t_0 \leq t < b_1$$

where c *is a positive constant,* W *and* W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) \int_{t_0}^{s} k(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (2.1), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.2) is hS, the solution $v = 0$ of (2.3) is hS. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.4) is hS. By Lemma 2.4, Lemma 2.5, and the nondecreasing property of the function h, we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| ds \\ &\leq c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)a(s)w(\frac{|y(s)|}{h(s)})ds \\ &\quad + \int_{t_0}^{t} c_2 h(t)b(s) \int_{t_0}^{s} k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.8, we obtain

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big],\ t_0 \le t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. This completes the proof. □

Remark 3.2. Letting $k(\tau) = 0$ in Theorem 3.1, we have the similar result as that of Theorem 3.2 in [8].

Theorem 3.3. *Let $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u, $u \le w(u)$, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that the solution $x = 0$ of (2.2) is hS with a nondecreasing function h and the perturbed term g in (2.1) satisfies*

$$|\Phi(t, s, y)g(t, y, Ty)| \le a(s)w(|y|) + b(s)|Ty|,\ t \ge t_0 \ge 0,$$

and

$$|Ty| \le \int_{t_0}^{t} k(s)|y(s)|ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, and $\int_{t_0}^{\infty} k(s)ds < \infty$. Then any solution $y(t) = y(t, t_0, y_0)$ of (2.1) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big],\ t_0 \le t < b_1.$$

where W and W^{-1} are the same functions as in Lemma 2.8, c is a positive constant, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (2.1), respectively. By Lemma 2.5, we obtain

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))g(s, y(s), Ty(s))|ds \\ &\le c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t}(a(s)w(|y(s)|) + b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau)ds \\ &\le c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} a(s)h(t)w(\frac{|y(s)|}{h(s)})ds \\ &\quad + \int_{t_0}^{t} b(s)\int_{t_0}^{s} h(t)k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau ds \end{aligned}$$

since h is nondecreasing. Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.8, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s}k(\tau)d\tau)ds\Big],\ t_0 \leq t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. Therefore, we obtain the result. □

REMARK 3.4. Letting $k(\tau) = 0$ in Theorem 3.3, we have the similar result as that of Theorem 3.1 in [8].

THEOREM 3.5. *Let $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondeacreasing in u, $u \leq w(u)$, and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $x = 0$ of (2.2) is an h-system with a positive continuous function h and g in (2.1) satisfies*

$$|g(t, y, Ty)| \leq a(t)w(|y(t)|) + b(t)|Ty(t)|,\ t \geq t_0,\ y \in \mathbb{R}^n$$

and

$$|Ty(t)| \leq \int_{t_0}^{t}k(s)|y(s)|ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with

$$\int_{t_0}^{\infty}\frac{1}{h(s)}\int_{t_0}^{s}(a(\tau)h(\tau) + b(\tau)\int_{t_0}^{\tau}h(r)k(r)dr)d\tau ds < \infty, \tag{3.1}$$

for all $t_0 \geq 0$, $\int_{t_0}^{\infty}a(s)ds < \infty$, $\int_{t_0}^{\infty}b(s)ds < \infty$, and $\int_{t_0}^{\infty}k(s)ds < \infty$, then any solution $y(t) = y(t, t_0, y_0)$ of (2.1) satisfies

$$|y(t)| \leq h(t)W^{-1}\Big[W(c)+\int_{t_0}^{t}\frac{c_2}{h(s)}\int_{t_0}^{s}(a(\tau)h(\tau)+b(\tau)\int_{t_0}^{\tau}h(r)k(r)dr)d\tau ds\Big],$$

$t_0 \leq t < b_1$, where W and W^{-1} are the same functions as in Lemma 2.8, c is a positive constant, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t}\frac{c_2}{h(s)}\int_{t_0}^{s}(a(\tau)h(\tau) + b(\tau)\int_{t_0}^{\tau}h(r)k(r)dr)d\tau ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (2.1), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.2) is hS, the solution $v = 0$ of (2.3) is hS. Therefore, by Theorem

2.7, the solution $z = 0$ of (2.4) is hS. By Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \int_{t_0}^{s} |g(\tau, y(\tau), Ty(\tau))| d\tau ds \\ &\le c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 \frac{h(t)}{h(s)} \int_{t_0}^{s} a(\tau) h(\tau) w(\frac{|y(\tau)|}{h(\tau)}) d\tau ds \\ &\quad + \int_{t_0}^{t} c_2 \frac{h(t)}{h(s)} \int_{t_0}^{s} b(\tau) \int_{t_0}^{\tau} h(r)k(r) \frac{|y(r)|}{h(r)} dr d\tau ds.\end{aligned}$$

Setting $u(t) = |y(t)| h(t)^{-1}$ and using Lemma 2.9, we obtain

$$\begin{aligned}|y(t)| \le h(t) W^{-1}\Big[W(c) &+ \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} (a(\tau)h(\tau) \\ &+ b(\tau) \int_{t_0}^{\tau} h(r)k(r) dr) d\tau ds\Big],\end{aligned}$$

$t_0 \le t < b_1$, where $c = c_1 |y_0| h(t_0)^{-1}$. Hence, the proof is complete. □

REMARK 3.6. Letting $k(\tau) = 0$ in Theorem 3.5, we have the similar result as that of Theorem 3.5 in [8].

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